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Number-Phase Quantization Scheme and the Quantum Effects of a Mesoscopic Electric Circuit at Finite Temperature

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Abstract For L-C circuit, a new quantized scheme has been proposed in the context of number-phase quantization. In this quantization scheme, the number n of the electric charge $q(q = en)$ is quantized as the charge number operator and the phase difference θ across the capacity is quantized as phase operator. Based on the scheme of number-phase quantization and the thermo field dynamics (TFD), the quantum fluctuations of the charge number and phase difference of a mesoscopic L-C circuit in the thermal vacuum state, the thermal coherent state and the thermal squeezed state have been studied. It is shown that these quantum fluctuations of the charge number and phase difference are related to not only the parameters of circuit, the squeezing parameter, but also the temperature in these quantum states. It is proven that the number-phase quantization scheme is very useful to tackle with quantization of some mesoscopic electric circuits and the quantum effects.

Keywords Quantum optics · Mesoscopic L-C circuit · Number-phase quantization · Thermal squeezed state · Quantum fluctuations

1 Introduction

According to the rapid development of nanophysics and nanoelectrionics, the quantum effects of mesoscopic circuits have brought much interest of physicists. Louisell first quantized a mesoscopic L-C circuit [1]. In his scheme, electric charge q is quantized as the coordinate operator and electric current I multiplied by L is quantized as the momentum operator, then the L-C circuit is considered as a quantum harmonic oscillator. Based on Louisell's quantization scheme, much meaningful work about the electric circuits has been done [2–5]. Recently, instead of using Louisell's quantization scheme, a new quantized scheme has been proposed for mesoscopic L-C circuit in the context of number-phase quantization [6–8]. In this number-phase quantization scheme, the charge number and phase difference across the capacity also make up a conjugate commutative relation.

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In this paper, based on the number-phase quantization scheme and the Umezawa-Takahashi thermo field dynamics (TFD) [9], we will study the quantum effects of mesoscopic L-C circuit in the thermal vacuum state, the thermal coherent state and the thermal squeezed. It is shown that quantum fluctuations of the charge number and phase difference are related to not only the parameters of circuit, the squeezing parameter, but also the temperature in these quantum states. In addition, we can obtain that the energy stored in the capacitance and the inductance of the mesoscopic L-C circuit at finite temperature are same to that bases on the Louisell's scheme [10]. These results provide support for the number-phase quantization scheme and have meaningful to study the quantum effects of mecosocpic electric circuit.

2 Number-Phase Quantization Scheme for Mesoscopic L-C Circuit

For a fundamental L-C circuit, considering the discreteness of electric charge, the electric charge can be expressed by $q = en$, where n is the number of electrons across the inductance. Based on [7, 8], the Lagrangian function of the system is

$$L = T - V = \frac{1}{2}LI^2 - \frac{1}{2C}q^2 = \frac{1}{2}Le^2\dot{n}^2 - \frac{1}{2C}e^2n^2, \tag{1}$$

where $T = \frac{1}{2}LI^2 = \frac{1}{2}L(\frac{\mathrm{d}q}{\mathrm{d}t})^2 = \frac{1}{2}Le^2\dot{n}^2$ is considered as kinetic energy stored in the inductance, $V = \frac{1}{2C}q^2 = \frac{1}{2C}e^2n^2$ is considered as potential energy stored in the capacitance. Based on the Hamilton canonical equation, the corresponding canonical momentum conjugated to n is

$$p = \frac{\partial L}{\partial \dot{n}} = Le^2\dot{n} = e\phi, \tag{2}$$

where ϕ is the magnetic flux caused by self-inductance. According to the Faraday theorem, we know that the variation of ϕ yields the voltage $V \equiv -\frac{\mathrm{d}\phi}{\mathrm{d}t}$, which is equal to the value of the voltage across the capacity. This voltage, from the quantum mechanical wave function's viewpoint, is related to the phase difference between two plates of the capacity within a time interval dt [11, 12]. Assuming the wave function at the two plates respectively are

$$\Psi_i = \phi_i \mathrm{e}^{\mathrm{i}E_i t/\hbar} = \phi_i \mathrm{e}^{\mathrm{i}\omega_i t} = \phi_i \mathrm{e}^{\mathrm{i}\theta_i}, \quad i = 1, 2, \tag{3}$$

noting

$$\mathrm{d}\theta_i = \frac{E_i}{\hbar}\mathrm{d}t, \qquad \mathrm{d}\theta = \mathrm{d}\theta_2 - \mathrm{d}\theta_1 = \frac{E_2 - E_1}{\hbar}\mathrm{d}t, \tag{4}$$

then the energy gap between two plates in the capacity is

$$E_1 - E_2 = -\hbar\frac{\mathrm{d}\theta}{\mathrm{d}t} = e\mathrm{V}. \tag{5}$$

So combining (2)–(5) and the Faraday theorem, the relation between the canonical momentum and phase difference across the capacity is

$$p = \hbar\theta. \tag{6}$$

Based on the number-phase quantization scheme, $\hbar\theta$ is quantized as the canonical momentum, while the charge number n is quantized as the canonical coordinate. It is clear that they satisfy the following commutative relation

$$[\hat{n}, \hbar\hat{\theta}] = \mathrm{i}\hbar \quad \text{or} \quad [\hat{n}, \hat{\theta}] = \mathrm{i}. \tag{7}$$

The uncertainty relation $\Delta n \Delta\theta \geq 1/2$, which means quantum fluctuation. Therefore, the operator Hamiltonian for a mesoscopic L-C circuit can be as

$$H = T + V \to \hat{H} = \frac{1}{2}\frac{\hbar^2\hat{\theta}^2}{e^2 L} + \frac{1}{2}L\omega^2 e^2 \hat{n}^2, \tag{8}$$

where $\omega = \sqrt{\frac{1}{LC}}$ the resonant frequency of the electric circuit. Thus the quantization of mesoscopic L-C circuit in the context of number-phase quantization scheme is realized.

According to the above new quantized scheme, we can introduce the following annihilation and creation operators from (8)

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{e^2 L\omega}{\hbar}} \cdot \hat{n} + \mathrm{i}\sqrt{\frac{\hbar}{e^2 L\omega}} \cdot \hat{\theta}\right), \tag{9}$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{e^2 L\omega}{\hbar}} \cdot \hat{n} - \mathrm{i}\sqrt{\frac{\hbar}{e^2 L\omega}} \cdot \hat{\theta}\right). \tag{10}$$

Based on the relation $[\hat{n}, \hat{\theta}] = \mathrm{i}$, it can be easy proved

$$[\hat{a}, \hat{a}^\dagger] = 1. \tag{11}$$

Therefore the operator Hamiltonian $\hat{H}$ can be rewritten as

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right). \tag{12}$$

So after number-phase quantized, the mesoscopic L-C circuit can be considered as a quantum harmonic oscillator. From (9) and (10), we deduce

$$\hat{n} = \sqrt{\frac{\hbar}{2e^2 L\omega}}(\hat{a} + \hat{a}^\dagger), \qquad \hat{\theta} = \frac{1}{\mathrm{i}}\sqrt{\frac{e^2 L\omega}{2\hbar}}(\hat{a} - \hat{a}^\dagger). \tag{13}$$

3 Quantum Effects of the Charge Number and Phase Difference at Finite Temperature

We introduce a tilde space besides the Hilbert space in the TFD theorem [9], the direct product space is made up of the above two spaces. Every operator and state in the Hilbert space has corresponding operator and state in the tilde space. $\hat{a}$ and $\hat{a}^\dagger$ have the corresponding operators $\tilde{\hat{a}}$ and $\tilde{\hat{a}}^\dagger$ which obey

$$[\tilde{\hat{a}}, \tilde{\hat{a}}^\dagger] = 1, \qquad [\tilde{\hat{a}}, \hat{a}] = [\tilde{\hat{a}}, \hat{a}^\dagger] = [\hat{a}, \tilde{\hat{a}}^\dagger] = [\hat{a}^\dagger, \tilde{\hat{a}}^\dagger] = 0. \tag{14}$$

The vacuum state $|0\tilde{0}\rangle$ is the usual product vacuum in a doubled Hilbert space, the thermal vacuum state can be obtained by thermo operator $T(\theta)$. That is

$$|0\rangle_T = T(\theta)|0\tilde{0}\rangle, \quad T(\theta) = \exp[-\theta(a\tilde{\hat{a}} - \hat{a}^\dagger\tilde{\hat{a}}^\dagger)], \tag{15}$$

where $|0\rangle_T$ denotes the thermal vacuum state, θ relates to count average thermal particle amount n_0, $n_0 = \sinh^2\theta$, while the relation between n_0 and temperature satisfies the Boson-Einstein distribution

$$n_0 = (\mathrm{e}^{\hbar\omega\beta} - 1)^{-1}, \tag{16}$$

where κ_B is Boltzmann's constant, $\beta = 1/\kappa_B T$.

In [13], the definition of the thermal coherent state is

$$|\alpha\rangle_T = D(\alpha)T(\theta)|0\tilde{0}\rangle, \quad D(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a}) \tag{17}$$

where $D(\alpha)$ is the standard displacement operator, α express the displacement parameter. And the definition of the thermal squeezed state is

$$|\alpha, z\rangle_T = D(\alpha)S(z)T(\theta)|0\tilde{0}\rangle, \quad S(z) = \exp\frac{1}{2}(z^*\hat{a}^2 - z\hat{a}^{\dagger 2}) \tag{18}$$

where $z = r\exp(\mathrm{i}\phi)$, r and ϕ express the squeezed parameter and squeezed angel, respectively.

Combining (11), (14), (15), (17), (18) and the Baker-Hausdorff formula $\mathrm{e}^B A \mathrm{e}^{-B} = A + [B, A] + \frac{1}{2!}[B, [B, A]] + \cdots$, we can obtain

$$\begin{aligned} T^\dagger(\theta)\hat{a}T(\theta) &= \hat{a}\cosh\theta + \tilde{\hat{a}}^\dagger\sinh\theta, \\ T^\dagger(\theta)\hat{a}^\dagger T(\theta) &= \hat{a}^\dagger\cosh\theta + \tilde{\hat{a}}\sinh\theta, \end{aligned} \tag{19}$$

$$\begin{aligned} D^\dagger(\alpha)\hat{a}D(\alpha) &= \hat{a} + \alpha, \\ D^\dagger(\alpha)\hat{a}^\dagger D(\alpha) &= \hat{a}^\dagger + \alpha^*, \end{aligned} \tag{20}$$

$$\begin{aligned} S^\dagger(z)\hat{a}S(z) &= \hat{a}\cosh r - \hat{a}^\dagger\exp(\mathrm{i}\phi)\sinh r, \\ S^\dagger(z)\hat{a}^\dagger S(z) &= \hat{a}^\dagger\cosh r - \hat{a}\exp(-\mathrm{i}\phi)\sinh r. \end{aligned} \tag{21}$$

According to (13) and (19)–(21), we can obtain the average, the mean square values of the charge number and phase difference in the thermal vacuum state, the thermal coherent state and the thermal squeezed state are respectively

$${}_T\langle 0|\hat{n}|0\rangle_T = 0, \qquad {}_T\langle 0|\hat{n}^2|0\rangle_T = \frac{\hbar}{2e^2L\omega}(2n_0 + 1), \tag{22}$$

$${}_T\langle 0|\hat{\theta}|0\rangle_T = 0, \qquad {}_T\langle 0|\hat{\theta}^2|0\rangle_T = \frac{e^2L\omega}{2\hbar}(2n_0 + 1), \tag{23}$$

$$\begin{aligned} {}_T\langle\alpha|\hat{n}|\alpha\rangle_T &= \sqrt{\frac{\hbar}{2e^2L\omega}}(\alpha + \alpha^*), \\ {}_T\langle\alpha|\hat{n}^2|\alpha\rangle_T &= \frac{\hbar}{2e^2L\omega}[(\alpha + \alpha^*)^2 + (2n_0 + 1)], \end{aligned} \tag{24}$$

$$
\begin{aligned}
{}_T\langle\alpha|\hat{\theta}|\alpha\rangle_T &= \frac{1}{\mathrm{i}}\sqrt{\frac{e^2L\omega}{2\hbar}}(\alpha-\alpha^*),\\
{}_T\langle\alpha|\hat{\theta}^2|\alpha\rangle_T &= \frac{e^2L\omega}{2\hbar}[-(\alpha-\alpha^*)^2+(2n_0+1)],
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
{}_T\langle\alpha,z|\hat{n}|\alpha,z\rangle_T &= \sqrt{\frac{\hbar}{2e^2L\omega}}(\alpha+\alpha^*),\\
{}_T\langle\alpha,z|\hat{n}^2|\alpha,z\rangle_T &= \frac{\hbar}{2e^2L\omega}\{(\alpha+\alpha^*)^2+(2n_0+1)[\cos(2r)-\cos\phi\sinh(2r)]\},
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
{}_T\langle\alpha,z|\hat{\theta}|\alpha,z\rangle_T &= \frac{1}{\mathrm{i}}\sqrt{\frac{e^2L\omega}{2\hbar}}(\alpha-\alpha^*),\\
{}_T\langle\alpha,z|\hat{\theta}^2|\alpha,z\rangle_T &= \frac{e^2L\omega}{2\hbar}\{-(\alpha-\alpha^*)+(2n_0+1)[\cosh(2r)+\cos\phi\sinh(2r)]\}.
\end{aligned}
\tag{27}
$$

From (18) and (19), the average values of the charge number and phase difference are equal to zero in the thermal vacuum state, while the mean square values are not equal to zero but related with the environment temperature and parameters of electric circuit. Then the quantum fluctuations of the charge number and phase difference are

$$
\langle(\Delta n)^2\rangle_T = \frac{\hbar}{2e^2L\omega}\frac{1+\mathrm{e}^{-\beta\hbar\omega}}{1-\mathrm{e}^{-\beta\hbar\omega}},\qquad \langle(\Delta\theta)^2\rangle_T = \frac{e^2L\omega}{2\hbar}\frac{1+\mathrm{e}^{-\beta\hbar\omega}}{1-\mathrm{e}^{-\beta\hbar\omega}}.
\tag{28}
$$

The corresponding uncertainty relation is

$$
\langle(\Delta n)\rangle_T\langle(\Delta\theta)\rangle_T = \frac{1}{2}\frac{1+\mathrm{e}^{-\beta\hbar\omega}}{1-\mathrm{e}^{-\beta\hbar\omega}}.
\tag{29}
$$

This result is consistent to that of reference [2] in the theorem. Base on operator Hamiltonian for mesoscopic L-C circuit (8), we can obtain the kinetic energy stored in the inductance from (23)

$$
{}_T\langle 0|\widehat{T}|0\rangle_T = {}_T\langle 0|\frac{1}{2}\frac{\hbar^2\hat{\theta}^2}{e^2L}|0\rangle_T = \frac{\hbar\omega}{4}\frac{1+\mathrm{e}^{-\beta\hbar\omega}}{1-\mathrm{e}^{-\beta\hbar\omega}}.
\tag{30}
$$

And the potential energy stored in the capacity can be obtain from (22)

$$
{}_T\langle 0|\hat{V}|0\rangle_T = {}_T\langle 0|\frac{1}{2}L\omega^2e^2\hat{n}^2|0\rangle_T = \frac{\hbar\omega}{4}\frac{1+\mathrm{e}^{-\beta\hbar\omega}}{1-\mathrm{e}^{-\beta\hbar\omega}}.
\tag{31}
$$

So that, the whole energy of the system is

$$
{}_T\langle 0|\hat{H}|0\rangle_T = {}_T\langle 0|\widehat{T}|0\rangle_T + {}_T\langle 0|\hat{V}|0\rangle_T = \frac{\hbar\omega}{2}\frac{1+\mathrm{e}^{-\beta\hbar\omega}}{1-\mathrm{e}^{-\beta\hbar\omega}}.
\tag{32}
$$

It is shown that these results deriving from the number-phase quantization scheme are same to that of reference [10], based on the Louisell's scheme.

From the (24) and (25), the average values, the mean square values of charge number and phase difference are not equal to zero in the thermal coherent state, which relate to the environment temperature, displacement parameter and the parameters of electric circuit. While the quantum fluctuations and the corresponding uncertainty relation of the charge number and phase difference are same to those in the thermal vacuum state.

In the thermal squeezed state, the average values of charge number and phase difference are related to the displacement parameter and the parameters of electric circuit, but not to the environment temperature. The mean square values are related to not only the displacement parameter, the parameters of electric circuit, the squeezed parameters, but also the environment temperature. The quantum fluctuations of the charge number and phase difference are respectively

$$\langle(\Delta n)^2\rangle_{DST} = \frac{\hbar}{2e^2L\omega}[\cos(2r) - \cos\phi\sinh(2r)]\frac{1+\mathrm{e}^{-\beta\hbar\omega}}{1-\mathrm{e}^{-\beta\hbar\omega}}, \tag{33}$$

$$\langle(\Delta\theta)^2\rangle_{DST} = \frac{e^2L\omega}{2\hbar}[\cosh(2r) + \cos\phi\sinh(2r)]\frac{1+\mathrm{e}^{-\beta\hbar\omega}}{1-\mathrm{e}^{-\beta\hbar\omega}}. \tag{34}$$

The corresponding uncertainty is

$$\langle(\Delta n)\rangle_{DST}\langle(\Delta\theta)\rangle_{DST} = \frac{1}{2}[\cosh^2(2r) - \cos^2\phi\sinh^2(2r)]^{1/2}\frac{1+\mathrm{e}^{-\beta\hbar\omega}}{1-\mathrm{e}^{-\beta\hbar\omega}}. \tag{35}$$

It is evident that the quantum fluctuations, the corresponding uncertainty of the charge number and phase difference are related to the displacement parameter, the parameters of electric circuit, the squeezed parameters and the environment temperature.

From (33)–(35), when $r \to 0$, then $\cosh 2r \to 1$ and $\sinh 2r \to 0$, we can obtain the quantum fluctuations and the uncertainty relation of the charge number and phase difference in the thermal vacuum state or thermal coherent state. When $r \to 0$ and $T \to 0$, we obtain the quantum fluctuations and the uncertainty relation of charge number and phase difference in the vacuum state

$$\langle(\Delta n)^2\rangle = \frac{\hbar}{2e^2L\omega}, \qquad \langle(\Delta\theta)^2\rangle = \frac{e^2L\omega}{2\hbar}, \tag{36}$$

$$\langle(\Delta n)\rangle\langle(\Delta\theta)\rangle = \frac{1}{2}. \tag{37}$$

These results in vacuum state just coincide with that of Louisell's in the theorem, and quantum fluctuation (37) embodies the commutative relation (7) between the charge number and phase difference of mesoscopic L-C circuit.

4 Conclusion

In summary, based on the number-phase quantized scheme and TFD theorem, we study the quantum effects of the charge number and phase difference of a mesoscopic L-C circuit in the thermal vacuum state, the thermal coherent state and the thermal squeezed state. These results are shown that the quantum fluctuations of the charge number and phase difference are related to not only the parameters of circuit, the squeezing parameter, but also the temperature in these quantum states. It provides that the number-phase quantized scheme is a complementary view of coordinate-momentum quantized scheme, which seems fresh. Also it is proven that the number-phase analysis is very useful to tackle with the quantization of some mesoscopic electric circuits and quantum effects of the corresponding physical quantities.

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